Chapter 10
Introduction to Finite Element Methods in Computational Fluid Dynamics

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10.1 Introduction

The finite element method (FEM) is a numerical technique for solving partial differential equations (PDE's). Its first essential characteristic is that the continuum field, or *domain*, is subdivided into cells, called *elements*, which form a grid. The elements (in 2D) have a triangular or a quadrilateral form and can be rectilinear or curved. The grid itself need not be structured. With *unstructured* grids and *curved cells*, complex geometries can be handled with ease. This important advantage of the method is not shared by the finite difference method (FDM) which needs a structured grid, which, however, can be curved. The finite volume method (FVM), on the other hand, has the same geometric flexibility as the FEM.

The second essential characteristic of the FEM is that the solution of the discrete problem is assumed a priori to have a prescribed form. The solution has to belong to a *function space*, which is built by varying function values in a given way, for instance linearly or quadratically, between values in nodal points. The nodal points, or *nodes*, are typical points of the elements such as vertices, mid-side points, mid-element points, etc. Due to this choice, the representation of the solution is strongly linked to the geometric representation of the domain. This link is, for instance, not as strong in the FVM.

The third essential characteristic is that a FEM does not look for the solution of the PDE itself, but looks for a solution of an integral form of the PDE. The most general integral form is obtained from a *weighted residual formulation*. By this formulation the method acquires the ability to naturally incorporate *differential type boundary conditions* and allows easily the construction of higher order accurate methods. The ease in obtaining higher order accuracy and the ease of implementation of boundary conditions form a second important advantage of the FEM. With respect to accuracy, the FEM is superior to the FVM, where higher order accurate formulations are quite complicated.

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The combination of the representation of the solution in a given function space, with the integral formulation treating rigorously the boundary conditions, gives to the method an extremely *strong and rigorous mathematical foundation*.

A final essential characteristic of the FEM is the modular way in which the discretization is obtained. The discrete equations are constructed from contributions on the element level which afterwards are *assembled*.

Historically, the finite element method originates from the field of structural mechanics. This has some remnants in the terminology. In structural mechanics, the partial differential formulation of a problem can be replaced by an equivalent *variational formulation*, i.e. the minimization of an energy integral over the domain. The variational formulation is a natural integral formulation for the FEM. In fluid dynamics, in general, a variational formulation is not possible. This makes it less obvious how to formulate a finite element method. The history of computational fluid dynamics (CFD) shows that every essential break-through has first been made in the context of the finite difference method or the finite volume method and that it always has taken considerable time, often much more than a decade, to incorporate the same idea into the finite element method. The history of CFD, on the other hand, also shows that, once a suitable FEM-formulation has been found, the FEM is almost exclusively used. This is due to the advantages with respect to the treatment of complex geometries and obtaining higher order accuracy.

The development of the finite element method in fluid dynamics is at present still far from ended. For the simplest problems such as potential flows, both compressible and incompressible, and incompressible Navier-Stokes flows at low Reynolds numbers, the finite element method is more or less full-grown, although new evolutions, certainly for Navier-Stokes problems, are still continuing. More complex problems like compressible flows governed by Euler- or Navier-Stokes equations or incompressible viscous flows at high Reynolds numbers still form an area of active research.

In this introductory text, the option is taken to explain the basic ingredients of the finite element method on a very simple, purely mathematical, problem and to give fluid dynamics illustrations in detail only for simple problems. For more complex problems, only a basic description is given with reference to further literature. Also in the explanation of the method, all mathematical aspects are systematically avoided. For the mathematical aspects, reference is made to further literature. This makes the text accessible for a reader with almost no knowledge of functional analysis and numerical analysis. For the fluid dynamics illustrations, the option has been taken to use only simple techniques, so that the detailed examples can be reproduced by the reader not really familiar with general computational fluid dynamics or even general fluid dynamics. This text therefore is to be seen as the absolute minimum introduction to the subject. The text is in no way complete and the author deliberately has taken the risk to be seen as naive by a more informed reader. A reference list is given for a deeper introduction. A reader beginning with computational fluid dynamics should be aware that a complete study of the finite element method may take considerable time and may necessitate, depending on background, a considerable effort. The method is much less intuitive than the finite difference method and

the finite volume method and requires a more fundamental attitude for mathematical formulations. This introductory text therefore is also meant to create some enthusiasm for the method by showing its power with simple examples and to justify in this way the need for further study. It is the conviction of the author that a practitioner of CFD, even if it is not his or her intention to use the FEM, should have at least a basic knowledge of the method. This is in particular useful with respect to the treatment of boundary conditions. Also one should consider that the impact of the FEM in CFD is already extremely important and that it probably will grow in the future.

10.2 Strong and Weak Formulations of a Boundary Value Problem

10.2.1 Strong Formulation

Consider as an example, the following simple one-dimensional boundary value problem, consisting of the *differential equation*

$$\frac{d}{dx}\left(\lambda\frac{du}{dx}\right) = f \quad \text{on} \quad 0 \le x \le X \tag{10.1}$$

and the *boundary conditions*

$$u(0) = u_0 \tag{10.2}$$

and

$$\lambda\frac{du}{dx}(X) = q \tag{10.3}$$

More generally, the differential equation is denoted by

$$a(u) = f \tag{10.4}$$

The *domain* to which it applies is denoted by Ω. The boundary condition of type (10.2) is called a *Dirichlet boundary condition*. More generally, it is denoted by

$$b_0(u) = g_0 \tag{10.5}$$

The boundary condition of type (10.3), which is formulated on the *flux* of the variable, is called a *Neumann boundary condition*. More generally, it is denoted by

$$b_1(u) = g_1 \tag{10.6}$$

The boundary of the domain Ω is denoted by Γ. The part to which the Dirichlet boundary condition applies is Γ_0 and the part to which the Neumann boundary condition applies is Γ_1.

The *boundary value problem* (10.1, 10.2 and 10.3) is said to be in its *strong form*, requiring the satisfaction of the differential equation (10.1) in all points of the domain Ω, the satisfaction of the Dirichlet boundary condition (10.2) in all points (here one) of the part of the boundary Γ_0 and the satisfaction of the Neumann boundary condition (10.3) in all points (here one) of the part of the boundary Γ_1.

One way of relaxing the requirements of the boundary value problem, notably the *finite difference way*, consists in requiring the approximate satisfaction of the differential equation and the boundary conditions in a finite number of points in the domain and at the boundary. These points usually are chosen to belong to a *mesh* with some form of regularity. For the one-dimensional domain, a typical mesh or grid is obtained by choosing equally spaced *grid points*, as shown on Fig. 10.1.

The *grid spacing* is denoted by Δx. Following standard finite difference methodology, du/dx is approximated in the mid-point of the interval (x_i, x_{i+1}) by

$$\left(\frac{du}{dx}\right)_{i+1/2} \approx \frac{u_{i+1} - u_i}{\Delta x} \tag{10.7}$$

Similarly, in the mid-point of the interval (x_{i-1}, x_i), the approximation is

$$\left(\frac{du}{dx}\right)_{i-1/2} \approx \frac{u_i - u_{i-1}}{\Delta x} \tag{10.8}$$

Using (10.7) and (10.8), (10.1) can be approximated by

$$\frac{\lambda_{i+1/2}(u_{i+1} - u_i) - \lambda_{i-1/2}(u_i - u_{i-1})}{\Delta x^2} = f_i \tag{10.9}$$

For constant λ, this simplifies to

$$\lambda \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} = f_i \tag{10.10}$$

The Dirichlet boundary condition (10.2) is simply

$$u_0 = u_0 \tag{10.11}$$

The Neumann boundary condition can be introduced by the *image point method*. In this method, a point outside the domain $(N + 1)$ is defined which afterwards is eliminated. The discretization of the differential equation (1) in the end point of the domain is given by (10.9) for $i = N$.

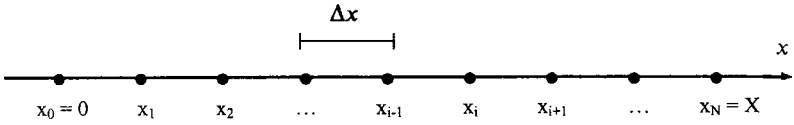

Fig. 10.1 Construction of a finite difference grid over the interval $0 \leq \times \leq X$

The discretization of the Neumann boundary condition (10.3) is

$$\frac{1}{2}\frac{\lambda_{N+1/2}(u_{N+1}-u_N)}{\Delta x}+\frac{1}{2}\frac{\lambda_{N-1/2}(u_N-u_{N-1})}{\Delta x}=q$$

Combination with the discretized differential equation gives

$$q-\lambda_{N-1/2}\frac{(u_N-u_{N-1})}{\Delta x}=\frac{1}{2}f_N\Delta x \tag{10.12}$$

The resulting discretization is of second order. By taking the Taylor expansion of (10.10), this is obvious (for constant λ) for points inside the domain. At the Neumann boundary, the Taylor expansion up to second order (for constant λ) gives

$$u_{N-1}\approx u_N-\Delta x\left(\frac{du}{dx}\right)_N+\frac{1}{2}\Delta x^2\left(\frac{d^2u}{dx^2}\right)_N$$

Using the Neumann boundary condition

$$\lambda\left(\frac{du}{dx}\right)_N=q$$

and the differential equation in node N

$$\lambda\left(\frac{d^2u}{dx^2}\right)_N=f_N$$

this becomes

$$u_{N-1}\approx u_N-\frac{\Delta x}{\lambda}q+\frac{1}{2}\frac{\Delta x^2}{\lambda}f_N$$

For constant λ, this equation is identical to (10.12).

The originally continuous boundary value problem is now replaced by a *discrete problem*, consisting of the solution of the *set of algebraic equations*

$$K\,U=F \tag{10.13}$$

where U is the vector consisting of the elements $(u_1, u_2, \ldots, u_N)$, K is a matrix given by (in the case λ is a constant)

$$K=\begin{bmatrix}2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ldots & & \\ & & -1 & 2 & -1\\ & & & -1 & 1\end{bmatrix}$$

and F is the right hand side, given by

$$F = \begin{bmatrix} u_0 - \dfrac{\Delta x^2}{\lambda} f_1 \\ -\dfrac{\Delta x^2}{\lambda} f_2 \\ \vdots \\ -\dfrac{\Delta x^2}{\lambda} f_{N-1} \\ \dfrac{\Delta x}{\lambda} q - \dfrac{\Delta x^2}{2\lambda} f_N \end{bmatrix}$$

The most typical feature of the finite difference method is that it only gives information about the function values at the grid points, but no information on the function values between these points.

10.2.2 Weighted Residual Formulation

The first basic ingredient of the finite element method is that an approximate solution is sought which belongs to some *finite dimension function space*. This function space is to be specified more in detail later. For the time being, we look for an approximate solution of the boundary value problem (10.1, 10.2 and 10.3) which has the form

$$\hat{u} = \psi + \sum_{k=1}^{N} \phi_k \, u_k \tag{10.14}$$

where ψ is a function which satisfies the boundary conditions (10.2) and (10.3). For the given problem, the construction of ψ is obvious. The functions ϕ_k are called *basis functions* or *shape functions*. Since the dimension of the function space $\Phi = \{\phi_k;\ k = 1,\ 2, \ldots,\ N\}$ is finite, in general, an expression of type (10.14) cannot satisfy the differential equation (10.1) in each point of the domain. This means that the approximate solution $\hat{u}$ cannot be identical with the exact solution u. Of course, the shape functions should be chosen so that by enriching the function space Φ, i.e. letting N grow, the approximation obtained by (10.14) becomes better. This means that the approximate solution *converges* to the exact solution. This is called the *completeness requirement* of the function space.

Since a function $\hat{u}$ given by (10.14) cannot satisfy the differential equation (10.1), upon substitution of (10.14) into (10.1), a *residual* is left:

$$r_\Omega = a(\hat{u}) - f \quad \text{in } \Omega \tag{10.15}$$

An approximate solution of the boundary value problem now is obtained by finding a way to make this residual small in some sense. In the finite element method this is done by requiring that an appropriate number of *weighted integrals* of the residual over Ω be zero:

$$\int_{\Omega} w_i r_\Omega d\Omega = 0; \quad i = 1,2,\ldots,N \tag{10.16}$$

where $W = \{w_i;\ i = 1,\ 2,\ldots,\ N\}$ is a set of *weighting functions*.

Obviously, the convergence requirement now also implies a requirement of completeness of the space of weighting functions, i.e. (10.16) should imply $r_\Omega \to 0$ for $N \to \infty$.

Clearly, with satisfaction of the completeness, for $N \to \infty$, the *weighted residual formulation* (10.16) for a function of form (10.14) is completely equivalent to the strong formulation of the problem (10.1, 10.2 and 10.3). An approximate solution then is obtained for N being finite.

10.2.3 Galerkin Formulation

Among the possible choices for the set of weighting functions, the following ones are the most obvious.

The weighting functions can be chosen to be Dirac-delta functions in N points. This choice means making the residual equal to zero in a number of chosen points. The method is called the *point collocation method*. Obviously, it has much in common with the finite difference methodology.

A second possible choice of weighting functions is given by

$$\begin{aligned} w_i &= 1 \quad \text{for } x_i \le x \le x_{i+1} \\ &= 0 \quad \text{for } x < x_i \text{ or } x > x_{i+1} \end{aligned}$$

The weighted residual statements (10.16) now require the integral of the residual to be zero on N subdomains. This method is called the *subdomain collocation method*. The finite volume method, in which not the differential form of the equation but the integral form of the equation is discretized, is a special form of this method.

The most popular choice for the weighting functions in the finite element method is the shape functions themselves:

$$w_i = \phi_i$$

This method is called the *Galerkin method*. Its meaning is that the residual is made to be orthogonal to the space of the shape functions.

To illustrate the Galerkin method, consider the boundary value problem (10.1–10.3) with constant λ. Then:

$$\psi = u_0 + \frac{q}{\lambda} x$$

Consider further as an example of (10.14) a Fourier-sine expansion of u:

$$\hat{u} = \psi + \sum_{k=1}^{N} u_k \sin \frac{\pi k' x}{X}, \quad \text{with } k' = k - 1/2$$

Then:

$$r_\Omega = -\lambda \sum_{k=1}^{N} u_k \left(\frac{\pi k'}{X}\right)^2 \sin\frac{\pi k' x}{X} - f$$

The Galerkin method then gives

$$\lambda \sum_{k=1}^{N} u_k \left(\frac{\pi k'}{X}\right)^2 \int_0^X \sin\frac{\pi k' x}{X} \sin\frac{\pi i' x}{X} dx = -\int_0^X \sin\frac{\pi i' x}{X} f\, dx$$

Then noting that

$$\int_0^X \sin\frac{\pi k' x}{X} \sin\frac{\pi i' x}{X} dx = \frac{X}{2} \quad \text{for } k' = i'$$

$$= 0 \quad \text{for } k' \neq i'$$

we find

$$u_i = -\frac{2X}{\lambda \pi^2 i'^2} \int_0^X f \sin\frac{\pi i' x}{X} dx$$

The foregoing method used to determine an approximate solution of the boundary value problem (10.1, 10.2 and 10.3) is not a finite element method, but a *spectral method*. The finite element method however has the same starting point.

Before going on with the study of the building blocks of the finite element method, we should remark that a fourth weighted residual statement exists on which finite element methods can be based. The *least squares formulation* is based on the minimization of the integral

$$\int_\Omega r_\Omega^2 d\Omega$$

10.2.4 Weak Formulation

In many problems, it is not practical to construct a function which satisfies the boundary conditions in order to arrive at an expression for the approximate solution, as is done in (14). More generally, an approximate solution can be expressed as

$$\hat{u} = \sum_{k=1}^{N} \phi_k u_k \tag{10.17}$$

Now the approximate solution not only has a residual with respect to the field equation (10.4), but also with respect to the boundary equations (10.5) and (10.6):

$$r_0 = b_0(\hat{u}) - g_0 \tag{10.18}$$

and

$$r_1 = b_1(\hat{u}) - g_1 \tag{10.19}$$

A weighted residual statement is now to be of the form

$$\int_{\Omega} w_i r d\Omega + \int_{\Gamma_0} w_i^0 r_0 d\Gamma + \int_{\Gamma_1} w_i^1 r_1 d\Gamma = 0 \tag{10.20}$$

This complicates the formulation since now additional weighting functions on the boundaries are to be chosen. Since the number of degrees of freedom of the approximate solution (10.17) is N, an equal number of independent weighting functions w_i can be chosen, while w_i^0 and w_i^1 are to depend on w_i. There is however a natural way to choose the dependent weighting functions on the boundary.

For the problem (10.1, 10.2 and 10.3), (10.20) becomes

$$\int_0^X w_i \left[\frac{d}{dx}\left(\lambda \frac{d\hat{u}}{dx}\right) - f \right] dx + w_i^0[\hat{u}(0) - u_0] + w_i^1 \left[\lambda \frac{d\hat{u}}{dx}(X) - q \right] = 0 \tag{10.21}$$

where the weighting functions at the boundary reduce to weighting factors w_i^0 and w_i^1.

By one integration by parts on the first term, (10.21) becomes

$$w_i \lambda \frac{d\hat{u}}{dx}\Bigg|_0^X - \int_0^X \lambda \frac{dw_i}{dx}\frac{d\hat{u}}{dx} dx - \int_0^X w_i f \, dx + w_i^0[\hat{u}(0) - u_0] + w_i^1 \left[\lambda \frac{d\hat{u}}{dx}(X) - q \right] = 0$$

This weighted residual statement is simplified by choosing the weighting factors on the Neumann boundary by

$$w_i^1 = -w_i(X)$$

The weighted residual statement then becomes

$$-\int_0^X \lambda \frac{dw_i}{dx}\frac{d\hat{u}}{dx} dx - w_i(0)\lambda \frac{d\hat{u}}{dx}(0) - \int_0^X w_i f \, dx + w_i^0[\hat{u}(0) - u_0] + w_i(X)q = 0$$

Furthermore, if the Dirichlet boundary condition can be imposed on the approximate solution, the weighting functions and the weighting factors can be chosen to be zero at the Dirichlet boundary, so that the weighted residual statement further simplifies to

$$-\int_0^X \lambda \frac{dw_i}{dx}\frac{d\hat{u}}{dx} dx - \int_0^X w_i f \, dx + w_i(X)q = 0 \tag{10.22}$$

subject to the Dirichlet boundary conditions

$$\hat{u}(0) = u_0 \quad w_i(0) = 0 \tag{10.23}$$

The weighted residual statement in form (10.22) is called the *weak formulation*.

The weak formulation (10.22 and 10.23) is not completely equivalent to the strong formulation (10.1, 10.2, 10.3), even not for $N \to \infty$. By the construction of the weak formulation, any solution of the strong formulation satisfies the weak formulation. The reverse, however, is not true. The weak formulation allows solutions which have a lower degree of regularity than required for the strong solution. This is the origin of the term *weak* and *strong*. For instance for the problem (10.1, 10.2, 10.3), the solution must have continuous first derivatives. We express this by saying that the *degree of regularity* is to be C^1. The corresponding weak formulation (10.22 and 10.23) only requires continuity of the function value itself. The necessary degree of regularity is here C^0. This means that functions with discontinuous first derivatives are allowed by (10.22). *We remark that this is precisely, certainly in fluid mechanics, what we want*! Indeed, in fluid mechanics, the governing equations are obtained from integral statements, i.e. conservation laws, requiring a lower degree of regularity than the partial differential equations which are obtained from these statements.

To conclude, we remark that the weak formulation (10.22), in case of sufficient regularity, through reverse integration by parts leads to a simplification of (10.21):

$$\int_0^X w_i \left[\frac{d}{dx}\left(\lambda \frac{d\hat{u}}{dx}\right) - f \right] dx - w_i(X)\left[\lambda \frac{d\hat{u}}{dx}(X) - q\right] = 0 \tag{10.24}$$

For an infinite number of degrees of freedom ($N \to \infty$), this implies exact satisfaction of the differential equation and the Neumann boundary condition.

In the weak formulation (10.22 and 10.23), the Neumann boundary condition need not be imposed in an explicit way to the solution. Boundary conditions of this type enter through the integration by parts in a natural way into the formulation. Therefore these boundary conditions are called *natural boundary conditions*. The boundary conditions which have to be imposed explicitly in the weak formulation are called *essential boundary conditions*.

10.2.5 Variational formulation

For elliptic self-adjoint boundary value problems, the weak formulation is equivalent to the minimization of the functional associated to the boundary value problem. Historically, this minimization formulation, or variational formulation, has played a big role in the development of the finite element method. Variational methods still have an important role in, for instance, structural mechanics. Also the variational formulation plays an important role in the mathematical theory of finite element

methods, for instance, with respect to questions on solvability and uniqueness. In this introductory text we do not enter these aspects of the finite element method and refer the reader to Refs. [5, 6].

10.2.6 Conclusion

The first basic ingredient of a finite element method generally is the *weak formulation of the boundary value problem*. Although, as discussed, other formulations are possible (more general weighted residual formulations and least squares formulations), a standard finite element method is based on a weak formulation. In this introductory text, we shall restrict ourselves to this formulation. If possible, the Galerkin approach is chosen with weighting functions equal to shape functions. We shall however see that sometimes modifications of this standard choice are necessary. The standard choice is denoted by the term Bubnov-Galerkin method. When modified weighting functions are used, the method is denoted by the term Petrov-Galerkin method (see later).

10.3 Piecewise Defined Shape Functions

10.3.1 The Finite Element Interpolation

A second basic ingredient of the finite element method is the *piecewise manner* in which the shape and weighting functions are constructed. The domain Ω is subdivided into *non-overlapping subdomains*, Ω_e, called *elements*, of simple geometrical form. For example, for the one dimensional domain shown in Fig. 10.1, an obvious choice for an element Ω_e is the interval $x_{e-1} \le x \le x_e$.

The integrals in the weak formulation (10.22) can be split into a sum of integrals over elements:

$$\int_{\Omega} (\,)\mathrm{d}\Omega = \sum_{e} \int_{\Omega_e} (\,)\mathrm{d}\Omega$$

Then, obviously, in the piecewise contributions to the integrals, it is computationally advantageous to have as many zero contributions as possible. This is achieved when the shape functions and weighting functions associated to some subscript are only non-zero in as few as possible elements associated to this subscript. Shape and weighting functions which are only non-zero in a small set of elements are said to have *compact support*.

In the finite element method, shape and weighting functions with compact support are constructed from an *interpolation problem* over the domain. For instance, a function $\hat{u}$ which is obtained through linear interpolation between function values u_k defined in the grid points of the grid of Fig. 10.1 can be written as